The (\(t\))-property of some classes of graphs

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Received 26 April 2007; received in revised form 15 December 2007; accepted 17 December 2007

Abstract

In this note, the (\(t\))-properties of five classes of graphs are studied. We prove that the classes of cographs and clique perfect graphs without isolated vertices satisfy the (2)-property and the (3)-property, but do not satisfy the (\(t\))-property for \(t \geq 4\). The (\(t\))-properties of the planar graphs and the perfect graphs are also studied. We obtain a necessary and sufficient condition for the restricted graph of index \(k\) to satisfy the (2)-property.

Keywords: Clique transversal number; (\(t\))-property

1. Introduction

We consider only finite, simple graphs \(G = (V, E)\) with \(|V| = n\) and \(|E| = m\).

A complete of a graph \(G\) is a complete subgraph of \(G\) and a clique of a graph \(G\) is a maximal complete of \(G\). A subset \(V'\) of \(V\) is called a clique transversal if it intersects with every clique of \(G\). The clique transversal number \(\tau_c(G)\) of a graph \(G\) is the minimum cardinality of a clique transversal of \(G\) [13]. For details, the reader may refer to [1,6,12].

The order \(n\) of \(G\) is an obvious upper bound for the clique transversal number. In an attempt to find graphs which admit a better upper bound, Tuza [13] introduced the concept of the (\(t\))-property. A class \(\mathcal{G}\) of graphs satisfies the (\(t\))-property if \(\tau_c(G) \leq \frac{n}{t}\) for every \(G \in \mathcal{G}\) where every edge of \(G\) is contained in a \(K_t \subseteq G\). Note that the (\(t\))-property does not imply the (\(t - 1\))-property.

It is known [7] that every chordal graph satisfies the (2)-property. In [13], it is proved that the (3)-property holds for chordal graphs; split graphs have the (4)-property, but do not have the (5)-property and hence the chordal graphs also do not have the (5)-property. It is proved [9] that the (4)-property does not hold for chordal graphs.

Motivated by the open problems mentioned in [7], we studied the (\(t\))-property for the cographs, the clique perfect graphs, the perfect graphs, the planar graphs and the restricted graphs of index \(k\). The cographs are a subclass of the perfect graphs [10] and also of the clique perfect graphs [12].

The (\(t\))-properties of the various classes of graphs which we studied in this paper are summarized in the following table.

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0012-365X/$ - see front matter © 2008 Published by Elsevier B.V.

Please cite this article in press as: S. Aparna Lakshmanan, A. Vijayakumar, The (\(t\))-property of some classes of graphs, Discrete Mathematics (2008), doi:10.1016/j.disc.2007.12.057
<table>
<thead>
<tr>
<th>Class</th>
<th>Satisfy (t)-property</th>
<th>Do not satisfy (t)-property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cographs</td>
<td>2, 3</td>
<td>≥ 4</td>
</tr>
<tr>
<td>Clique perfect graphs</td>
<td>2, 3</td>
<td>≥ 4</td>
</tr>
<tr>
<td>Planar graphs</td>
<td>--</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>Perfect graphs</td>
<td>--</td>
<td>≥ 2</td>
</tr>
</tbody>
</table>

All graph theoretic terminology and notation not mentioned here are from [2].

2. The (t)-property

2.1. Cographs and clique perfect graphs

A graph which does not have $P_4$ - the path on four vertices - as an induced subgraph is called a cograph. The join of two graphs $G$ and $H$ is defined as the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv, where u \in V(G) and v \in V(H)\}$.

Cographs [5] can also be recursively defined as follows:

(1) $K_1$ is a cograph;
(2) if $G$ is a cograph, so is its complement $\overline{G}$; and
(3) if $G$ and $H$ are cographs, so is their join $G \vee H$.

A clique independent set is a subset of pairwise disjoint cliques of $G$. The clique independence number $\alpha_c(G)$ of a graph $G$ is the maximum cardinality of a clique independent set of $G$. Clearly, $\alpha_c(G)$ is a lower bound for $\tau_c(G)$. A graph for which this lower bound is attained for all its induced subgraphs also is called a clique perfect graph [3, 11].

The class of cographs is clique perfect [12]. A characterization of clique perfect graphs by means of a list of minimal forbidden subgraphs is still an open problem.

**Lemma 1.** If $G = G_1 \vee G_2$ then $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$.

**Proof.** Any clique in $G$ is of the form $C_1 \vee C_2$ where $C_1$ is a clique in $G_1$ and $C_2$ is a clique in $G_2$. If $V'$ is a clique transversal of $G_1$ (or $G_2$), then any clique of $G$ which contains a clique of $G_1$ (or $G_2$) is covered by $V'$ and hence $V'$ is a clique transversal of $G$ also.

Now, let $V'$ be a clique transversal of $G$. If possible assume that $V'$ does not cover cliques of $G_1$ and $G_2$. Let $H_1$ and $H_2$ be the cliques of $G_1$ and $G_2$ respectively which are not covered by $V'$. Then $H_1 \vee H_2$ is a clique of $G$ which is not covered by $V'$, which is a contradiction. Hence $V'$ contains a clique transversal of $G_1$ or $G_2$.

Therefore, $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$.

**Lemma 2.** The class of all cographs without isolated vertices does not satisfy the (t)-property for $t \geq 4$.

**Proof.** The proof is by construction.

**Case 1:** $t = 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_2) \vee (3K_1 \cup K_2)$ and $G_2 = (3K_1 \cup K_2)$. Then $n = 15$, $t = 4$ and $\tau_c(G) = 4$ which implies that $\frac{n}{t} < \tau_c(G)$.

**Case 2:** $t > 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_{t-3}) \vee (3K_1 \cup K_{t-3})$ and $G_2 = (3K_2 \cup K_{t-2})$.

Then $n(G) = 3t + 4$ and $\tau_c(G) = 4$.

Every edge in $G_1$ lies in a complete of size $t$ in $G$ since $G_2$ contains a clique of size $t - 2$. Every edge in $G_2$ lies in a complete of size $t - 2$ in $G$ since $G_1$ contains a clique of size $t - 2$. An edge with one end vertex in $G_1$ and the other end vertex in $G_2$ lies in a complete of size $t - 2$ since every vertex in $G_1$ lies in a complete of size $t - 2$ and every vertex of $G_2$ lies in a complete of size 2. Hence $G$ is a cograph in which every edge lies in a clique of size $t$.

Also, $\frac{n}{t} = 3 + \frac{4}{t}$

Therefore, $\frac{n}{t} < \tau_c(G)$ for $t > 4$. 

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Theorem 3. The class of clique perfect graphs without isolated vertices satisfies the \( (t) \)-property for \( t = 2 \) and \( 3 \) and does not satisfy the \( (t) \)-property for \( t \geq 4 \).

Proof. Let \( G \) be a clique perfect graph in which every edge lies in a complete of size \( t \). \( G \) being clique perfect, \( \tau_c(G) = \alpha_c(G) \).

Case 1: \( t = 2 \)

Since \( G \) is without isolated vertices \( \alpha_c(G) \leq \frac{n}{2} \). So \( \tau_c(G) = \alpha_c(G) \leq \frac{n}{2} \) and hence the class of clique perfect graphs satisfies the \( (2) \)-property.

Case 2: \( t = 3 \)

Every edge of \( G \) lies in a clique of size 3. So, the size of the smallest clique of \( G \) is 3. Therefore, \( \alpha_c(G) \leq \frac{n}{3} \) and \( \tau_c(G) = \alpha_c(G) \leq \frac{n}{3} \).

Case 3: \( t \geq 4 \)

The class of cographs is a subclass of clique perfect graphs. So by Lemma 2, the claim follows.

Corollary 4. The class of cographs without isolated vertices satisfies the \( (t) \)-property for \( t = 2 \) and \( 3 \). Moreover, for the class of connected cographs without isolated vertices, \( \tau_c(G) \) is maximum if and only if \( G \) is the complete bipartite graph \( K_{\frac{n}{2}, \frac{n}{2}} \).

Proof. Since the class of cographs is a subclass of clique perfect graphs, it satisfies the \( (t) \)-property for \( t = 2 \) and \( 3 \).

Since the class of cographs satisfy the \( (2) \)-property and \( \tau_c(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n}{2} \), the maximum value of \( \tau_c(G) \) is \( \frac{n}{2} \).

Conversely, let \( G \) be a connected cograph with \( \tau_c(G) = \frac{n}{2} \). Since \( G \) is a connected cograph, \( G = G_1 \cup G_2 \). Therefore, \( \tau_c(G) = \min(\tau_c(G_1), \tau_c(G_2)) \). But, \( \tau_c(G_1) \) and \( \tau_c(G_2) \) cannot exceed the numbers of vertices in \( G_1 \) and \( G_2 \) respectively and hence the number of vertices in \( G_1 \) and \( G_2 \) must be \( \frac{n}{2} \). Again, since \( \tau_c(G) = \frac{n}{2} \) all these vertices must be isolated. Therefore, \( G = K_{\frac{n}{2}, \frac{n}{2}} \).

Corollary 5. For the class of clique perfect graphs without isolated vertices, \( \tau_c(G) \) is maximum if and only if there exists a perfect matching in \( G \) in which no edge lies in a triangle.

Proof. The class of clique perfect graphs without isolated vertices satisfies the \( (2) \)-property. Therefore, the maximum value that \( \tau_c(G) \) can obtain is \( \frac{n}{2} \). Let \( G \) be a clique perfect graph with \( \tau_c(G) = \frac{n}{2} \). \( G \) being clique perfect, \( \alpha_c(G) = \tau_c(G) = \frac{n}{2} \). Since each clique must have at least two vertices and there are \( \frac{n}{2} \) independent cliques, the cliques are of size exactly 2. Again, this independent set of \( \frac{n}{2} \) cliques forms a perfect matching of \( G \) and a clique being maximal complete, the edges of this perfect matching do not lie in triangles.

Conversely, if there exists a perfect matching in which no edge lies in a triangle, the edges of this perfect matching form an independent set of cliques with cardinality \( \frac{n}{2} \). Therefore, \( \alpha_c(G) \geq \frac{n}{2} \). But, \( \alpha_c(G) \leq \tau_c(G) \leq \frac{n}{2} \) and therefore \( \tau_c(G) = \frac{n}{2} \).

2.2. Planar graphs

It is known that a graph \( G \) is planar if and only if it has no subgraph homeomorphic to \( K_5 \) or \( K_{3,3} \).

Theorem 6. The class of planar graphs does not satisfy the \( (t) \)-property for \( t = 2, 3 \) and \( 4 \) and \( G_t \) is empty for \( t \geq 5 \).

Proof. Every odd cycle is a planar graph and \( \tau_c(C_{2k+1}) = k + 1 > \frac{2k+1}{2} \). Clearly, odd cycles belong to \( G_2 \) and hence the class of planar graphs does not satisfy the \( (2) \)-property.

The graph in Fig. 1 is planar and every edge lies in a triangle. Here, \( n = 8 \) and the clique transversal number is 3 which is greater than \( \frac{n}{2} \) and hence planar graphs do not satisfy the \( (3) \)-property. Also, the graph in Fig. 2 is planar and every edge lies in a \( K_4 \). Here, \( n = 15 \) and the clique transversal number is 4 which is greater than \( \frac{n}{2} \) and hence planar graphs do not satisfy the \( (4) \)-property.

Since \( K_5 \) is a forbidden subgraph for planar graphs, there is no planar graph \( G \) such that all its edges lie in a \( K_t \) for \( t \geq 5 \). Hence, the theorem.
2.3. Perfect graphs

A graph $G$ is perfect if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$, where $\chi(H)$ is the chromatic number and $\omega(H)$ is the clique number of $H$ [10]. By the celebrated strong perfect graph theorem [4], a graph is perfect if and only if it has no odd hole or odd anti-hole as an induced subgraph.

**Theorem 7.** The class of perfect graphs does not satisfy the $(t)$-property for any $t \geq 2$.

**Proof.** Let $G$ be the cycle of length $3k$, say $v_1v_2 \ldots v_{3k}v_1$ where $k > 2$ is odd, in which the vertices $v_1, v_4, \ldots, v_{3k-2}$ are all adjacent to each other. Then $G$ is perfect and $\tau_c(G) = \left\lceil \frac{3k}{2} \right\rceil > \frac{3k}{2}$, since $3k$ is odd. Therefore the class of perfect graphs does not satisfy the $(2)$-property.

Now, the class of perfect graphs does not satisfy the $(3)$-property since $C_8$ is a perfect graph in which every edge lies in a triangle and $\tau_c(C_8) = 3 > \frac{8}{3}$.

Since the cographs are a subclass of perfect graphs [5], by Lemma 2, the class of perfect graphs also does not satisfy the $(t)$-property for $t \geq 4$.

2.4. Trellied graph of index $k$

For a graph $G$, $T_k(G)$ the trellied graph of index $k$ is the graph obtained from $G$ by adding $k$ copies of $K_2$ for each edge $uv$ of $G$ and joining $u$ and $v$ to the respective end vertices of each $K_2$ [8]. The vertex cover number of a graph $G$, denoted by $\beta(G)$, is the minimum number of vertices required to cover all the edges of $G$.

**Lemma 8.** For any graph $G$ without isolated vertices, $\tau_c(T_k(G)) = km + \beta(G)$.

**Proof.** We shall prove the theorem for the case $k = 1$.

Let $V' = \{v_1, v_2, \ldots, v_7\}$ be a vertex cover of $G$. The cliques of $T_1(G)$ are precisely the cliques of $G$ together with the three $K_2$s formed corresponding to each edge of $G$. Corresponding to each edge $uv$ of $G$ choose the vertex which corresponds to $u$ of the corresponding $K_2$, if $u$ is not present in $V'$. If $u$ is present in $V'$, then choose the vertex corresponding to $v$, irrespective of whether $v$ is present in $V'$ or not. Let this new collection together with $V'$ be $V''$. Then $V''$ is a clique transversal of $T_1(G)$ of cardinality $m + \beta(G)$. Therefore, $\tau_c(T_1(G)) \leq m + \beta(G)$. 

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Let \( V' = \{v_1, v_2, \ldots, v_t \} \), where \( t = \tau_c(T_1(G)) \), be a clique transversal of \( T_1(G) \). Let \( uv \) be an edge in \( G \) and let \( u'v' \) be the \( K_2 \) introduced in \( T_1(G) \) corresponding to this \( K_2 \). At least one vertex from \( \{u', v'\} \), say \( u' \), must be present in \( V' \), since \( V' \) is a clique transversal and \( u'v' \) is a clique of \( T_1(G) \). Remove \( u' \) from \( V' \). If \( V' \) contains \( v' \) also then replace \( v' \) by \( v \). If \( v' \notin V' \) then \( v \in V' \), since \( V' \) is a clique transversal and \( uv' \) is a clique of \( T_1(G) \). In either case, one vertex \( v' \) of the edge \( uv \) is present in the new collection. Repeat the process for each edge in \( G \) to get \( V'' \). Clearly, \( V'' \) is a vertex cover of \( G \) with cardinality \( \tau_c(T_1(G)) - m \). Hence, \( \beta(G) \leq \tau_c(T_1(G)) - m \). Thus, \( \tau_c(T_1(G)) = m + \beta(G) \).

By a similar argument we can prove that \( \tau_c(T_k(G)) = km + \beta(G) \).

**Notation.** For a given class \( \mathcal{G} \) of graphs, let \( T_k(\mathcal{G}) = \{ T_k(G) : G \in \mathcal{G} \} \).

**Theorem 9.** The class \( T_k(\mathcal{G}) \) satisfies the \((2)\)-property if and only if \( \beta(G) \leq \frac{n}{2} \forall G \in \mathcal{G} \) and \( T_k(G) \) is empty for \( t \geq 3 \).

**Proof.** Let \( G \in \mathcal{G} \), \( n(T_k(G)) = n + km \) and by Lemma 8, \( \tau_c(T_k(G)) = km + \beta(G) \). Therefore,

\[
\tau_c(T_k(G)) \leq \frac{n(T_k(G))}{2} (=) km + \beta(G) \leq \frac{n + 2km}{2} (=) \beta(G) \leq \frac{n}{2}.
\]

Hence, \( T_k(\mathcal{G}) \) satisfies \((2)\)-property if and only if \( \beta(G) \leq \frac{n}{2} \forall G \in \mathcal{G} \).

If \( G \) contains at least one edge then \( T_k(G) \) has a clique of size 2 and hence \( T_k(G) \) is empty for \( t \geq 3 \).

**Acknowledgements**

The authors thank the referees for their suggestions for the improvement of this paper.

**References**